MATH 137 PRACTICE FINAL EXAM

Notes:

1. Answer all questions in the space provided. You may use the last page as additional space for solutions. Clearly mark this if you do.
2. Your grade will be influenced by how clearly you express your ideas and how well you organize your solutions. Show all details to get full marks. Numerical answers should be in exact values (no approximations). For example, $\frac{\sqrt{3}}{2}$ is acceptable, 0.8660 is not.
3. There are a total of ?? possible points.
4. Check that your exam has ?? pages, including the cover page.
5. DO NOT write on the Crowdmark QR code at the top of the pages or your exam will not be scanned (and will receive a grade of zero).
6. Use a dark pen or pencil.

(MC) Answer the following multiple choice questions by writing either **a**, **b**, **c**, or **d** in the box to the right of the question. Note there is only one correct answer for each question.

1. $\lim_{x\to 3} \ln|x-3| =$
   (a) 0.
   (b) ∞.
   (c) $-\infty$.
   (d) None of the above.

2. If f is continuous on $[a,b]$ and differentiable on (a,b), then
   (a) for any $x_1, x_2 \in (a,b)$ where $x_1 < x_2$, there exists $c \in (x_1, x_2)$ so that $f'(c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$.
   (b) $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$.
   (c) $f'(x)$ is continuous on (a,b).
   (d) None of the above.

3. $\lim_{x\to 0} \dfrac{\cos(x) + \sin(x) - 1}{x} =$
   (a) -1.
   (b) 0.
   (c) 2.
   (d) None of the above.

4. If $\lim_{x\to a} f(x) = L \in \mathbb{R}$ and $\{x_n\}$ is a sequence such that $x_n \to a$ as $n \to \infty$, and $x_n \neq a$ for all $n \in \mathbb{N}$, then
   (a) $\lim_{n\to\infty} f(x_n)$ does not exist.
   (b) f is continuous at $x = a$.
   (c) $\lim_{n\to\infty} f(x_n) = L$.
   (d) None of the above.

5. For a function f and $a \in \mathbb{R}$, if $f''(a) = 0$ and $f'(a) = 0$, then
   (a) $x = a$ is a point of inflection for f.
   (b) $x = a$ is a critical point of f.
   (c) f cannot have a local maximum at $x = a$.
   (d) None of the above.

(TF) True/False, answer in the box below the question by writing TRUE or FALSE.

6. TRUE or FALSE: For $a \in \mathbb{R}$, $|x - a| \le 1$ defines a closed interval of length 1.

7. TRUE or FALSE: $f(x) = 3x^4 + 2x - 1$ has a root on $[0,1]$.

8. TRUE or FALSE: If $f'(x) = \cos(x)$ then $f(x) = \sin(x)$.

9. TRUE or FALSE: Let $a_n = f(n)$ where f is a continuous function defined on $\mathbb{R}$. If $\lim_{n\to\infty} a_n = L$ then $\lim_{x\to\infty} f(x) = L$.

10. TRUE or FALSE: If f is not differentiable at $x = a \in \mathbb{R}$, then for $k \in \mathbb{R}$, $g(x) = f(x) + k$ is not differentiable at $x = a$.

(SA) Short answer questions, marks only awarded for a correct final answer, you do not need to show any work. **Clearly indicate your final answer.**

1. For $f(x) = \ln(e + x)$, find $L_0^f(x)$.

2. If f is a differentiable function such that $f(0) = 1$ and $f'(x) \in [1, 5]$ for all $x \in \mathbb{R}$, use the Bounded Derivative Theorem to write down an interval that $f(3)$ must lie in.

3. Give an example of a differentiable function f that is concave up everywhere, but $f''(0)$ does not exist.

4. Give an example of a function f that is differentiable on $(0, 1)$, both $f(0)$ and $f(1)$ are defined, but the Mean Value Theorem cannot be applied to f.

5. If $f(3) = 1$ and $f'(3) = \pi$, find $(f^{-1})'(1)$.

(LA) The remaining questions are long answer questions, please show all of your work.

1. Find each of the following sequence limits, if they exist. If they do not exist, prove it.

   (a) $\lim_{n\to\infty} \frac{\sin(n\pi)}{\sin(n)}$

   (b) $\lim_{n\to\infty} \frac{\sin(n)}{n^2+1}$

   (c) $\lim_{n\to\infty} \frac{n^3+n+1}{3n^3+n^2}$

2. Prove that if $\{a_n\}$, and $\{b_n\}$ are sequences such that $\{a_n\}$ is bounded and $b_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} (a_n b_n) = 0$.

3. For each of the following functions, compute $f'(x)$ using any method. You do not need to simplify your answers.

   (a) $f(x) = x^2 e^x \ln(x)$

   (b) $f(x) = \tan(\cos(x))$.

4. (a) Find y' if $\ln(x) + \ln(y) = xy$.

   (b) Find $\frac{dy}{dx}$ if $y = (\sin(x))^{\ln(x)}$ for $0 < x \leq \pi$.

5. For $f(x) = (x-1)|x+2| - 3$, determine all global extrema on the inteval $[-3, 0]$, if they exist.

6. (a) State the Intermediate Value Theorem for a function f.

   (b) Find an interval of length at most 1 that contains a root of $f(x) = x^3 + 3x + 1$.

   (c) Using $x_1 = 0$, perform two iterations of Newton's Method to find x_2 and x_3 to approximate the root of $f(x) = x^3 + 3x + 1$.

7. Let $f(x) = \ln(x^2 + 1)$.

   (a) Determine the intervals of increase/decrease for f.

   (b) Determine the intervals of concavity for f.

8. Prove that if f is a differentiable function with no critical points, then it can have at most one real root.

9. In each case, compute the limit using any method.

   (a) $\lim_{x\to 1^+} (\ln(x))^{x-1}$

   (b) $\lim_{x\to 0^+} \left(\sqrt{x}\right)^{\frac{1}{3\sqrt{x}}}$

   (c) $\lim_{x\to 0^+} \left(1+\sqrt{x}\right)^{\frac{1}{3\sqrt{x}}}$

10. Find values of a and b so that f is differentiable everywhere, where

$$f(x) = \begin{cases} \sin(ax) \text{ if } x \geq 0 \\ x^2 + 2x + b \text{ if } x < 0 \end{cases} .$$

11. (a) Prove that if $f'(x) = g'(x)$ for all x in some open interval I, then there exists $k \in \mathbb{R}$ so that $f(x) = g(x) + k$ for all $x \in I$.

    (b) Use part (a) to prove that if $f'(x) - g'(x) = 2x$ on I, then $f(x) = g(x) + x^2 + k$ for all $x \in I$for some $k \in \mathbb{R}$.

12. Consider the function $f(x) = \ln(1 + x)$.

    (a) Find the second-degree Taylor polynomial for f centred at $x = 0$, $T_{2,0}(x)$.

    (b) Use $T_{2,0}$ to approximate $\ln(2)$.

    (c) Use Taylor's Theorem to write down what $f(x) - T_{2,0}(x)$ is equal to (in terms of x and c) for $x > 0$.

    (d) Find an upper bound on the error in your approximation in part (b).

    (e) Is the estimate in part (b) an over or under estimate?

    (f) Give an interval that $\ln(2)$ must lie in, be as specific as possible.

13. Sketch the graph of $f(x)$, where

$$f(x) = \frac{(x-1)(x-4)}{(x-2)^2}, \qquad f'(x) = \frac{x+2}{(x-2)^3}, \qquad f''(x) = \frac{-2(x+4)}{(x-2)^4}.$$

Use this page for your work, on the next page you will summarize your findings and draw your graph. **Marks will be awarded to the next page only**. On your graph, label any intercepts, critical points, points of inflection, and asymptotes. In your summary, all points should include both x- and y-coordinates.

Summary:

Intercepts	**Asymptotes**	**Critical Points**	**Inflection Points**

The domain of f is:

Intervals where f is increasing:

Intervals where f is decreasing:

Intervals where f is concave up:

Intervals where f is concave down: